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Existence Principles for Nonlinear Resonant Operator and Integral Equations

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Abstract—Features of resonant operator equations are discussed, and existence results presented for nonlinear resonant integral equations. © 1998 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

In this paper, we discuss resonant operator equations and present existence results for nonlinear resonant integral equations.

We first discuss the resonant operator equation

$$y(t) = \lambda_m Ly(t) + Ny(t), \qquad \text{a.e. } t \in [0,T]. \tag{1.1}$$

We assume that $L : L^2[0,T] \to L^2[0,T]$ is a linear, completely continuous, self-adjoint, nonnegative operator; $N : L^2[0,T] \to L^2[0,T]$ is possibly nonlinear and λ_m is the m^{th} eigenvalue of L. Nonresonant operator equations of the form

$$y(t) = \gamma Ly(t) + Ny(t), \qquad \text{a.e. } t \in [0,T], \tag{1.2}$$

where L and N are as described above, are discussed in the literature [4]. Of course, the difference between (1.1) and (1.2) is that in (1.2) it is assumed that γ is not an eigenvalue of L, and one can readily apply the Fredholm Alternative to obtain an existence theory. The problems which arise due to the presence of λ_m in (1.1) are considered in Section 2, and we also discuss some aspects of an existence result for (1.1).

Our main interest in this paper is in nonlinear resonant integral equations, in particular equations of the form

$$y(t) = \lambda_m \int_0^T k(t,s)y(s)\,ds + \int_0^T k(t,s)f(s,y(s))\,ds \tag{1.3}$$

defined on $[0,T]$. Having alluded to the large role that the nonlinearity $N : L^2[0,T] \to L^2[0,T]$ plays in providing us with an existence result for (1.1) in Section 2, the integral equation (1.3)

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX gives us an excellent opportunity to study this role in detail. In Section 3, we illustrate the conditions that $k : [0,T] \times [0,T] \to \mathbf{R}$ and $f : [0,T] \times \mathbf{R} \to \mathbf{R}$ must satisfy in order to guarantee us first, a solution $y \in L^2[0,T]$, and second, a solution $y \in C[0,T]$ of (1.3).

We conclude the introduction by stating a definition and some well-known results from the literature that will be used throughout the paper.

DEFINITION 1.1. *Let I be an interval in $\mathbf{R}$. A function $f : I \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function if the following conditions hold:*

(i) *the map $t \to f(t,y)$ is measurable for all $y \in \mathbf{R}$,*
(ii) *the map $y \to f(t,y)$ is continuous for almost all $t \in I$.*

THEOREM 1.1. FREDHOLM ALTERNATIVE. *(See [3, p. 443].) Let $L : Y \to Y$ be a linear, completely continuous operator on a normed space Y, and let $\lambda \in \mathbf{R}$. Then*

$$y - \lambda L y = x \tag{1.4}$$

has a solution $y \in Y$ for every $x \in Y$, if and only if the homogeneous equation

$$y - \lambda L y = 0$$

has only the trivial solution $y = 0$. In this case, the solution of (1.4) is unique and $I - \lambda L : Y \to Y$ has a bounded, linear inverse.

THEOREM 1.2. NONLINEAR ALTERNATIVE. *Let C be a convex subset of a normed linear space E, and let U be an open subset of C, with $p^\star \in U$. Then every compact, continuous map $\tilde{N} : \overline{U} \to C$ has at least one of the following two properties:*

(i) *$\tilde{N}$ has a fixed point,*
(ii) *there is an $x \in \partial U$, with $x = (1-\delta)p^\star + \delta \tilde{N} x$ for some $0 < \delta < 1$.*

THEOREM 1.3. *Let H be a nonzero Hilbert space, and suppose that L is a bounded, self-adjoint, linear operator on H. Then at least one of $\pm\|L\|$ belong to $\sigma(L)$ and $\|L\| = \sup\{|\mu| : \mu \in \sigma(L)\}$, (here $Ly = \mu y$).*

2. RESONANT OPERATOR EQUATIONS

In this section, we discuss the resonant operator equation

$$y(t) = \lambda_m L y(t) + N y(t), \qquad \text{a.e. } t \in [0,T], \tag{2.1}$$

where λ_m will be described below. We assume throughout the section that

$$\begin{gathered} L : L^2[0,T] \to L^2[0,T] \text{ is a linear, completely continuous, self-adjoint,} \\ \text{nonnegative } \left(\langle Ly, y\rangle \ge 0, \text{ for all } y \in L^2[0,T]\right) \text{ operator, with} \\ L(u+iv) = Lu + iLv \text{ (here } u, v \text{ are real-valued) and } Lu, Lv \text{ real-valued} \end{gathered} \tag{2.2}$$

and

$$N : L^2[0,T] \to L^2[0,T] \text{ is a continuous and completely continuous operator} \tag{2.3}$$

are true. Since $L : L^2[0,T] \to L^2[0,T]$ is a linear, completely continuous, self-adjoint operator on an infinite dimensional separable Hilbert space, we know from the spectral theory of such operators [9] that L has a countably infinite number of real eigenvalues λ_i (we say that λ has an eigenvalue of L is there exists a nonzero $\psi \in L^2[0,T]$ such that $\lambda L \psi = \psi$), with corresponding (real-valued) eigenfunctions ψ_i which may be chosen to form an orthonormal basis for $L^2[0,T]$.

In addition, since L is a nonnegative operator, we see that $\lambda_i > 0$, for all i. We can arrange the eigenvalues of L such that

$$0 < \lambda_0 \le \lambda_1 \le \dots,$$

listed to multiplicity (see [2]). For simplicity, we will assume that

$$\begin{aligned}&\text{the consecutive eigenvalues } \lambda_m < \lambda_{m+1} \text{ of } L \text{ have multiplicity one, and the}\\ &\text{corresponding eigenfunctions } \psi_m \neq 0,\ \psi_{m+1} \neq 0 \text{ on a set of positive measure}\end{aligned} \tag{2.4}$$

holds also.

An existence result for the nonresonant counterpart of (2.1), namely

$$y(t) = \gamma Ly(t) + Ny(t), \qquad \text{a.e. } t \in [0,T], \tag{2.5}$$

where L and N are as described in (2.2) and (2.3), respectively, and γ is not an eigenvalue of L, is discussed in the literature [4]. In brief, rewriting (2.5) as

$$Ty(t) = Ny(t), \qquad \text{a.e. } t \in [0,T], \tag{2.6}$$

where

$$Ty(t) := (I - \gamma L)\, y(t), \qquad \text{a.e. } t \in [0,T],$$

one can see using the Fredholm Alternative (Theorem 1.1) that $T : L^2[0,T] \to L^2[0,T]$ has a bounded, linear inverse $T^{-1} : L^2[0,T] \to L^2[0,T]$. This fact enables one to write (2.6) as

$$y(t) = T^{-1}Ny(t), \qquad \text{a.e. } t \in [0,T]$$

and an existence principle which guarantees a solution $y \in L^2[0,T]$ of (2.5) then follows easily from the Nonlinear Alternative (Theorem 1.2). (We refer the reader to Theorem 2.2 and Theorem 2.3 of [4] for details.) For completion, we include the result here.

THEOREM 2.1. *Suppose that $L : L^2[0,T] \to L^2[0,T]$ satisfies (2.2) and (2.4), and $N : L^2[0,T] \to L^2[0,T]$ satisfies (2.3). In addition, suppose that there exists a constant $M > 0$, independent of δ, such that $\|y\|_2 \neq M$ for any solution $y \in L^2[0,T]$ of*

$$(I - \gamma L)y(t) = \delta Ny(t), \qquad \text{a.e. } t \in [0,T]$$

for each $\delta \in (0,1)$. Then (2.5) has at least one solution $y \in L^2[0,T]$.

REMARK 2.1. In general, if $L : L^2[0,T] \to X$, where $X = L^p[0,T]$, $p \ge 2$, or $X = C[0,T]$ is a linear, completely continuous operator, and γ is not an eigenvalue of L, then from the Fredholm Alternative we have that $I - \gamma L : X \to X$ has a bounded, linear inverse $(I - \gamma L)^{-1} : X \to X$.

If in addition to the above, L is also self-adjoint and nonnegative, then one can easily verify that the operator $(I - \gamma L)^{-1}L : X \to X$ is in turn, bounded, linear, and self-adjoint with eigenvalues $\lambda_i - \gamma$ and corresponding eigenfunctions ψ_i. Also from Theorem 1.3, we have for $\lambda_m < \gamma < \lambda_{m+1}$, that

$$\left\|(I - \gamma L)^{-1}L\right\| = \frac{1}{\overline{\gamma}},$$

where $\overline{\gamma} = \min\{\gamma - \lambda_m, \lambda_{m+1} - \gamma\}$.

In the spirit of the technique described above for the nonresonant, operator equation, we might write (2.1) as

$$T_{\lambda_m}y(t) = Ny(t), \qquad \text{a.e. } t \in [0,T],$$

where $T_{\lambda_m} : L^2[0,T] \to L^2[0,T]$ is defined by

$$T_{\lambda_m}y(t) := (I - \lambda_m L)y(t), \qquad \text{a.e. } t \in [0,T].$$

Of course the problem is immediately apparent—the operator T_{λ_m} is not invertible.

An alternative technique is to choose $\lambda_m < \gamma < \lambda_{m+1}$ and rewrite (2.1) as

$$(I - \gamma L)y(t) = (\lambda_m - \gamma)Ly(t) + Ny(t), \qquad \text{a.e. } t \in [0,T]. \tag{2.7}$$

Since $(\lambda_m - \gamma)L + N : L^2[0,T] \to L^2[0,T]$ is a continuous and completely continuous operator, and γ is not an eigenvalue of L, then the following existence principle for (2.1) is an immediate consequence of Theorem 2.1.

THEOREM 2.2. *Suppose that (2.2)–(2.4) hold and $\lambda_m < \gamma < \lambda_{m+1}$. In addition, suppose that there exists a constant $M > 0$, independent of δ, such that $\|y\|_2 \neq M$ for any solution $y \in L^2[0,T]$ of*

$$(I - \gamma L)y(t) = \delta\left[(\lambda_m - \gamma)Ly(t) + Ny(t)\right], \qquad \text{a.e. } t \in [0,T] \tag{2.8}$$

for each $\delta \in (0,1)$. Then (2.1) has at least one solution $y \in L^2[0,T]$.

On attempting to prove the existence of a constant $M > 0$, independent of δ, such that any solution $y \in L^2[0,T]$ of (2.8) is in fact bounded by M, (and thus write an existence result for (2.1)), we find that the nonlinear operator $N : L^2[0,T] \to L^2[0,T]$ plays a much more significant role here than it did in the nonresonant case. The conditions that N is required to satisfy seem strange and unnatural, even though in our applications to resonant integral equations this is not the case. Therefore, rather than present an existence result for (2.1) and have the integral equation result follow as a consequence, we choose to present an existence result directly for the resonant integral equation case (we refer the reader to Section 3). However, we will prove the following result which is "half-way towards being an existence result" for (2.1), and which will be used in the next section. We require the following notation.

NOTATION. Note that $L^2[0,T] = \Omega \oplus \Omega^\perp$ where $\Omega = \operatorname{span}\{\psi_1, \dots, \psi_m\}$. Let

$$u = \sum_{i=0}^{m} c_i\psi_i, \quad v = \sum_{i=m+1}^{\infty} c_i\psi_i, \quad y_m = c_m\psi_m \quad \text{and} \quad \tilde{y} = y - y_m, \qquad \text{where } c_i = \langle y, \psi_i\rangle . \tag{2.9}$$

Also note that $y = u + v = \tilde{y} + y_m$.

THEOREM 2.3. *Suppose that (2.2)–(2.4) and $\lambda_m < \gamma < \lambda_{m+1}$ hold, in addition to*

$$\begin{gathered}\text{there exist constants } A_1,\ A_2 \geq 0 \text{ such that for all } y \in L^2[0,T],\\ \text{and some } 0 \leq \alpha < 1,\ \|Ny\|_2 \leq A_1 + A_2\|y\|_2^\alpha.\end{gathered} \tag{2.10}$$

Then there exist constants B_1, $B_2 \geq 0$, independent of δ, such that

$$\|\tilde{y}\|_2 \leq B_1 + B_2\|y_m\|_2^\alpha \tag{2.11}$$

for any solution $y \in L^2[0,T]$ of (2.8), for each $\delta \in (0,1)$, (here $\tilde{y}$ and y_m are as described in (2.9)).

PROOF. Let $y \in L^2[0,T]$ be any solution of (2.8). Since γ is not an eigenvalue of L, we have from the Fredholm Alternative (Theorem 1.1) that $(I - \gamma L) : L^2[0,T] \to L^2[0,T]$ has a bounded, linear inverse, thus enabling us to write (2.8) as

$$T_\delta y(t) = \delta(I - \gamma L)^{-1}Ny(t), \qquad \text{a.e. } t \in [0,T], \tag{2.12}$$

where $T_\delta : L^2[0,T] \to L^2[0,T]$ is given by

$$T_\delta y(t) := \left(I - \delta(\lambda_m - \gamma)(I - \gamma L)^{-1}L\right) y(t), \qquad \text{a.e. } t \in [0,T]. \tag{2.13}$$

We claim that

$$\begin{gathered}\text{there exists } \epsilon > 0 \text{ such that for any } \delta \in (0,1)\\ \text{and any } y \in L^2[0,T],\ \ \epsilon\|\tilde{y}\|_2 \leq \|T_\delta y\|_2.\end{gathered} \tag{2.14}$$

To see this, recall from Remark 2.1 that $(I - \gamma L)^{-1}L : L^2[0,T] \to L^2[0,T]$ has eigenvalues $\lambda_i - \gamma$ with corresponding eigenfunctions ψ_i. Therefore,

$$\begin{aligned}\|T_\delta y\|_2^2 &= \left\|\left(I - \delta(\lambda_m - \gamma)(I - \gamma L)^{-1}L\right) y\right\|_2^2\\ &= \left\|\sum_{i=0}^{\infty} c_i\psi_i - \delta(\lambda_m - \gamma)\sum_{i=0}^{\infty}\frac{c_i\psi_i}{\lambda_i - \gamma}\right\|_2^2\\ &= \left\|\sum_{i=0}^{\infty} c_i\psi_i\left[1 - \delta\frac{(\lambda_m - \gamma)}{\lambda_i - \gamma}\right]\right\|_2^2\\ &= \sum_{i=0}^{\infty} c_i^2\left[1 - \delta\frac{(\lambda_m - \gamma)}{\lambda_i - \gamma}\right]^2.\end{aligned}$$

For all $\delta \in [0,1]$, we note that

$$1 \geq 1 - \delta\frac{(\lambda_m - \gamma)}{\lambda_i - \gamma} \geq 1 - \delta\frac{(\lambda_m - \gamma)}{\lambda_{m-1} - \gamma} \geq 1 - \frac{(\lambda_m - \gamma)}{\lambda_{m-1} - \gamma} > 0, \qquad \text{for } i < m,$$

$$1 \geq 1 - \delta\frac{(\lambda_m - \gamma)}{\lambda_i - \gamma} = 1 - \delta \geq 0, \qquad \text{for } i = m,$$

and

$$1 - \delta\frac{(\lambda_m - \gamma)}{\lambda_i - \gamma} \geq 1, \qquad \text{for } i > m.$$

These inequalities imply that

$$\begin{aligned}
\|T_\delta y\|_2^2 &= \sum_{i=0}^{\infty} c_i^2 \left[1 - \delta\frac{(\lambda_m - \gamma)}{\lambda_i - \gamma}\right]^2 \\
&\geq \left[1 - \delta\frac{(\lambda_m - \gamma)}{\lambda_{m-1} - \gamma}\right]^2 \sum_{i=0}^{m-1} c_i^2 + (1-\delta)^2 c_m^2 + \sum_{i=m+1}^{\infty} c_i^2 \\
&\geq \left[1 - \delta\frac{(\lambda_m - \gamma)}{\lambda_{m-1} - \gamma}\right]^2 \sum_{i=0}^{m-1} c_i^2 + \sum_{i=m+1}^{\infty} c_i^2 \\
&\geq \left[1 - \frac{(\lambda_m - \gamma)}{\lambda_{m-1} - \gamma}\right]^2 \sum_{\substack{i=0 \\ i \neq m}}^{\infty} c_i^2 = \epsilon^2 \|\tilde{y}\|_2^2,
\end{aligned}$$

where

$$\epsilon = 1 - \frac{(\lambda_m - \gamma)}{\lambda_{m-1} - \gamma}.$$

Thus the claim is proved and in fact we have explicitly found an $\epsilon > 0$ that satisfies (2.14).

REMARK 2.2. Note that

$$\begin{gathered}
\text{for each } \delta \in [0,1), \text{ there exists } \epsilon_\delta > 0 \text{ such that} \\
\text{for all } y \in L^2[0,T], \ \epsilon_\delta \|y\|_2 \leq \|T_\delta y\|_2
\end{gathered} \tag{2.15}$$

is true. In fact (2.15) can be proved using the Fredholm Alternative (Theorem 1.1), by showing that for each $\delta \in [0,1)$, $T_\delta y = 0$ implies that $y = 0$ (similar to [4, Theorem 2.2]). However, from the above analysis we can find an ϵ_δ explicitly for each $\delta \in [0,1)$.

To see this note from above that for $\delta \in [0,1]$,

$$\|T_\delta y\|_2^2 \geq \min\left\{\left[1 - \delta\frac{(\lambda_m - \gamma)}{\lambda_{m-1} - \gamma}\right]^2, \ (1-\delta)^2, \ 1\right\} \sum_{i=0}^{\infty} c_i^2.$$

If

$$\epsilon_\delta^2 = \min\left\{\left[1 - \delta\frac{(\lambda_m - \gamma)}{\lambda_{m-1} - \gamma}\right]^2, \ (1-\delta)^2, \ 1\right\},$$

then for $\delta \neq 1$, we have explicitly found an $\epsilon_\delta > 0$ that satisfies (2.15).

Now from (2.14), (2.12), (2.10) and the fact that $(I - \gamma L)^{-1} L : L^2[0,T] \to L^2[0,T]$ is a bounded, linear operator, we have that there exists an $\epsilon > 0$ such that for all $\delta \in (0,1)$

$$\epsilon \|\tilde{y}\|_2 \leq \|T_\delta y\|_2 \leq \left\|(I - \gamma L)^{-1} L\right\| \, \|Ny\|_2 \leq \left\|(I - \gamma L)^{-1} L\right\| \, \left(A_1 + A_2 \|y\|_2^\alpha\right). \tag{2.16}$$

In Remark 2.1, we showed that

$$\left\|(I - \gamma L)^{-1} L\right\| = \frac{1}{\overline{\gamma}}, \qquad \text{where } \overline{\gamma} = \min\{\gamma - \lambda_m, \ \lambda_{m+1} - \gamma\}.$$

In addition, since

$$\|y\|_2 = \left(\|\tilde{y}\|_2^2 + \|y_m\|_2^2\right)^{1/2}, \tag{2.17}$$

we obtain, using the fact that $(a+b)^{1/r} \le 2^{(r-1/r)}(a^{1/r}+b^{1/r})$, $a,b \ge 0$, $r \ge 1$,

$$\|y\|_2^\alpha = \left(\|\tilde{y}\|_2^2 + \|y_m\|_2^2\right)^{\alpha/2} \le 2^{(1-(\alpha/2))}\left(\|\tilde{y}\|_2^\alpha + \|y_m\|_2^\alpha\right).$$

Therefore, from (2.16) we now have

$$\|\tilde{y}\|_2 \le \frac{A_1}{\epsilon\overline{\gamma}} + \frac{A_2\, 2^{(1-(\alpha/2))}}{\epsilon\overline{\gamma}}\left(\|\tilde{y}\|_2^\alpha + \|y_m\|_2^\alpha\right). \tag{2.18}$$

But since $0 \le \alpha < 1$, we see from (2.18) that there exist constants $B_1, B_2 \ge 0$ such that

$$\|\tilde{y}\|_2 \le B_1 + B_2\|y_m\|_2^\alpha,$$

and the theorem is proved. ∎

3. EXISTENCE RESULTS FOR RESONANT OPERATOR EQUATIONS

In this section, we consider the nonlinear resonant integral equations

$$y(t) = \lambda_m \int_0^T k(t,s)y(s)\,ds + \int_0^T k(t,s)f(s,y(s))\,ds, \qquad \text{a.e. } t \in [0,T] \tag{3.1}$$

and

$$y(t) = \lambda_m \int_0^T k(t,s)y(s)\,ds + \int_0^T k(t,s)f(s,y(s))\,ds, \qquad t \in [0,T], \tag{3.2}$$

and present existence results which guarantee that (3.1) and (3.2) have a solution $y \in L^2[0,T]$ and $y \in C[0,T]$, respectively. Of course both equations can be written in the operator form of the previous section

$$y(t) = \lambda_m Ly(t) + Ny(t), \tag{3.3}$$

defined on $[0,T]$, by defining

$$Ly(t) := Ky(t) := \int_0^T k(t,s)y(s)\,ds, \tag{3.4}$$

$$Fy(t) := f(t,y(t)), \tag{3.5}$$

and

$$Ny(t) := KFy(t) = \int_0^T k(t,s)f(s,y(s))\,ds \tag{3.6}$$

on $[0,T]$. Therefore, we can apply the operator results of Section 2 to considerably lessen the work required here.

In particular, one can verify ([4, Theorem 3.1]) that if $k : [0,T] \times [0,T] \to \mathbf{R}$ satisfies

$$(t,s) \mapsto k(t,s) \text{ is measurable, and } \int_0^T \int_0^T |k(t,s)|^2\,dt\,ds < \infty, \tag{3.7}$$

$$k(t,s) = k(s,t) \text{ almost everywhere on } [0,T] \times [0,T] \tag{3.8}$$

and

$$\langle Ky, y\rangle \ge 0 \text{ for any } y \in L^2[0,T] \text{ (where } K \text{ is as defined in (3.4))}, \tag{3.9}$$

and $f : [0,T] \times \mathbf{R} \to \mathbf{R}$ satisfies

$$\begin{gathered} f(t,y) \text{ is a Carathéodory function (see Definition 1.1)} \\ \text{and } f(t,y) \in L^2[0,T] \text{ for } y \in L^2[0,T], \end{gathered} \tag{3.10}$$

then $L = K : L^2[0,T] \to L^2[0,T]$ and $N = KF : L^2[0,T] \to L^2[0,T]$ satisfy (2.2) and (2.3), respectively. For simplicity, we also assume that the kernel k in (3.4) is such that

$$\begin{gathered} \text{the consecutive eigenvalues } \lambda_m < \lambda_{m+1} \text{ of } K \text{ (as defined in (3.4))} \\ \text{have multiplicity one, and the corresponding eigenfunctions} \\ \psi_m \neq 0, \ \psi_{m+1} \neq 0 \text{ on a set of positive measure} \end{gathered} \tag{3.11}$$

holds, and thus $L = K : L^2[0,T] \to L^2[0,T]$ also satisfies (2.4). It now follows immediately from Theorem 2.2 that if $k : [0,T] \times [0,T] \to \mathbf{R}$ and $f : [0,T] \times \mathbf{R} \to \mathbf{R}$ are as described above, then (3.1) has a solution $y \in L^2[0,T]$ provided that we can prove the existence of a constant $M > 0$, independent of δ, such that $\|y\|_2 \leq M$ for any solution $y \in L^2[0,T]$ of

$$(I - \gamma K)y(t) = \delta \left[(\lambda_m - \gamma)Ky(t) + KFy(t)\right], \qquad \text{a.e. } t \in [0,T], \tag{3.12}$$

for each $\delta \in (0,1)$ and some $\lambda_m < \gamma < \lambda_{m+1}$. We will tackle the problem of proving the existence of such an M in two stages.

STAGE 1. We place a growth condition on $f : [0,T] \times \mathbf{R} \to \mathbf{R}$, say

$$|f(t,y)| \leq \phi_1(t) + \phi_2(t)|y|^\alpha, \quad \text{a.e. } t \in [0,T] \text{ where } 0 \leq \alpha < 1 \text{ and } \phi_1, \phi_2^{(1/1-\alpha)} \in L^2[0,T], \tag{3.13}$$

and then use the operator result, Theorem 2.3, to show that any solution $y \in L^2[0,T]$ of (3.12) satisfies

$$\begin{gathered} \text{there exist constants } B_1, B_2 \geq 0, \text{ independent of } \delta, \\ \text{such that } \|\tilde{y}\|_2 \leq B_1 + B_2\|y_m\|_2^\alpha. \end{gathered} \tag{3.14}$$

We state the result as the following theorem.

THEOREM 3.1. *Suppose $k : [0,T] \times [0,T] \to \mathbf{R}$ satisfies (3.7)–(3.9) and (3.11), and $f : [0,T] \times \mathbf{R} \to \mathbf{R}$ satisfies (3.10) and (3.13), then any solution $y \in L^2[0,T]$ of (3.12) satisfies (3.14).*

PROOF. For any $y \in L^2[0,T]$, using (3.13) and the fact that $\|K\| = 1/\lambda_0$, we see that

$$\begin{aligned} \|Ny\|_2^2 &= \|KFy\|_2^2 \leq \|K\|^2\|Fy\|_2^2 \\ &= \frac{1}{\lambda_0^2}\int_0^T \left(\phi_1(t) + \phi_2(t)|y(t)|^\alpha\right)^2 dt \\ &\leq \frac{1}{\lambda_0^2}\int_0^T \left(2\phi_1^2(t) + 2\phi_2^2(t)|y(t)|^{2\alpha}\right) dt \\ &\leq \frac{2}{\lambda_0^2}\left(\|\phi_1\|_2^2 + \left\|\phi_2^{(1/1-\alpha)}\right\|_2^{2(1-\alpha)} \|y\|_2^{2\alpha}\right), \end{aligned}$$

therefore, there exist constants A_1, $A_2 \geq 0$ such that

$$\|Ny\|_2 \leq A_1 + A_2\|y\|_2^\alpha.$$

Since L and N as defined in (3.4) and (3.6), respectively, satisfy (2.2)–(2.4), the result now follows immediately from Theorem 2.3. ∎

STAGE 2. We now claim that for a solution $y \in L^2[0,T]$ of (3.12), condition (3.14) implies that there exists a constant $M > 0$, independent of δ, such that $\|y\|_2 \leq M$.

Suppose that the claim is false. Then there exists a sequence (δ_n) in (0,1), and a sequence (y_n) in $L^2[0,T]$ such that

$$(I-\gamma K)y_n(t) = \delta_n\left[(\lambda_m-\gamma)Ky_n(t) + KFy_n(t)\right], \qquad \text{a.e. } t\in[0,T], \tag{3.15}$$

or equivalently,

$$(I-\lambda_m K)y_n(t) = (1-\delta_n)(\gamma-\lambda_m)Ky_n(t) + \delta_n KFy_n(t), \qquad \text{a.e. } t\in[0,T] \tag{3.16}$$

and

$$\|y_n\|_2 \to \infty, \qquad \text{as } n\to\infty. \tag{3.17}$$

NOTATION. In a similar fashion to (2.9), let

$$y_{mn} = c_{mn}\psi_m \quad \text{and} \quad \tilde{y_n} = y_n - y_{mn}, \qquad \text{where } c_{in} = \langle y_n, \psi_i\rangle .$$

From (2.17), (3.14), and (3.17), we also see that

$$\|y_{mn}\|_2 \to \infty, \qquad \text{as } n\to\infty. \tag{3.18}$$

Define for almost every $t\in[0,T]$,

$$r_n(t) = \frac{y_n(t)}{\|y_{mn}\|_2}, \quad \tilde{r_n}(t) = \frac{\tilde{y_n}(t)}{\|y_{mn}\|_2}, \quad \text{and} \quad r_{mn}(t) = \frac{y_{mn}(t)}{\|y_{mn}\|_2}.$$

Now since $0\le\alpha<1$, from (3.14) and (3.18) we obtain

$$\|\tilde{r_n}\|_2 = \frac{\|\tilde{y_n}\|_2}{\|y_{mn}\|_2} \le \frac{B_1 + B_2\|y_{mn}\|_2^\alpha}{\|y_{mn}\|_2} \to 0, \qquad \text{as } n\to\infty,$$

and therefore,

$$\tilde{r_n} \to 0 \text{ in } L^2[0,T], \qquad \text{as } n\to\infty. \tag{3.19}$$

Also since

$$r_{mn}(t) = \frac{y_{mn}(t)}{\|y_{mn}\|_2} = \frac{c_{mn}\psi_m(t)}{|c_{mn}|},$$

we have that there exists a subsequence S_1 of $\{1,2,\dots\}$ such that

$$r_{mn}(t) = A\psi_m(t), \qquad \text{a.e. } t\in[0,T], \quad \text{for all } n\in S_1, \text{ where } A \text{ equals } +1 \text{ or } -1. \tag{3.20}$$

Finally, we we see that

$$r_n \to A\psi_m \text{ in } L^2[0,T], \qquad \text{as } n\to\infty, \quad n\in S_1 \tag{3.21}$$

is true, since for $n\in S_1$, (3.19) and (3.20) imply

$$\|r_n - A\psi_m\|_2 = \|r_n - r_{mn}\|_2 = \|\tilde{r_n}\|_2 \to 0, \qquad \text{as } n\to\infty.$$

In fact from a standard result in Measure Theory, we have that there exists a subsequence S_2 of S_1 such that

$$r_n \to A\psi_m \text{ pointwise for,} \qquad \text{a.e. } t\in[0,T], \quad \text{as } n\to\infty, \quad n\in S_2. \tag{3.22}$$

Now multiplying (3.16) by r_{mn}/δ_n and integrating from 0 to T yields

$$0 = \frac{(1-\delta_n)(\gamma-\lambda_m)}{\delta_n\lambda_m}c_{mn}^2 + \langle KFy_n, r_{mn}\rangle,$$

which implies from (3.20) that for $n \in S_1$

$$0 \geq \langle KFy_n, r_{mn} \rangle = \langle KFy_n, A\psi_m \rangle . \tag{3.23}$$

To get a contradiction, we place additional assumptions on $f : [0,T] \times \mathbf{R} \to \mathbf{R}$ such that

$$\liminf_{n\to\infty} \langle KFy_n, A\psi_m \rangle > 0. \tag{3.24}$$

This will then contradict (3.23).

REMARK 3.1. Returning momentarily to the resonant operator equation (2.1) discussed in Section 2, it is clear from the above analysis that if, in addition to the conditions described in Theorem 2.3, we had a condition of the form

$$\liminf_{n\to\infty} \langle Ny_n, A\psi_m \rangle > 0, \quad \text{where } n \in S_2 \text{ and the } y_n \text{ are as described above,} \tag{3.25}$$

holding, then the existence of a solution $y \in L^2[0,T]$ of (2.1) would follow. However, we shirked from formally stating this result in Section 2 since, as we shall now illustrate, a condition of the form (3.25) follows more naturally when $N = KF : L^2[0,T] \to L^2[0,T]$ is the nonlinear integral operator as described in (3.6).

Since K is self-adjoint and has positive eigenvalues, (3.24) is true if

$$\liminf_{n\to\infty} \langle Fy_n, A\psi_m \rangle > 0 \tag{3.26}$$

holds. We will consider two cases here. The first case occurs when $\alpha = 0$ in (3.13), the second when $\alpha = 0$ or $1 > \alpha = \alpha_1/\alpha_2$, where α_1 is even and α_2 is odd.

CASE 1. Suppose $\alpha = 0$ in (3.13). Then since

$$|f(t, y_n(t))| \leq \phi_1(t), \qquad \text{a.e. } t \in [0,T],$$

we can apply Fatou's Lemma to obtain

$$\begin{aligned} \liminf_{n\to\infty} \langle Fy_n, A\psi_m \rangle &= \liminf_{n\to\infty} \int_0^T f(t, y_n(t)) A\psi_m(t)\, dt \\ &\geq \int_{I^+} \liminf_{n\to\infty} f(t, y_n(t)) A\psi_m(t)\, dt + \int_{I^-} \liminf_{n\to\infty} f(t, y_n(t)) A\psi_m(t)\, dt. \end{aligned} \tag{3.27}$$

Here

$$I^+ = \{t \in [0,T] = I : A\psi_m(t) > 0\} \quad \text{and} \quad I^- = \{t \in [0,T] : A\psi_m(t) < 0\}, \tag{3.28}$$

and $n \to \infty$ in S_2. Let (t_n) be any sequence. Recall that $\liminf s\, t_n = s \liminf t_n$ if s is a positive real number, whereas $\liminf s\, t_n = s \limsup t_n$ if s is a negative real number. Using these facts in (3.27) yields, for $n \in S_2$,

$$\liminf_{n\to\infty} \langle Fy_n, A\psi_m \rangle \geq \int_{I^+} A\psi_m(t) \liminf_{n\to\infty} f(t, y_n(t))\, dt + \int_{I^-} A\psi_m(t) \limsup_{n\to\infty} f(t, y_n(t))\, dt. \tag{3.29}$$

Recall condition (3.22). Let $M \subset [0,T]$ be the set of measure zero such that $r_n(t) \not\to A\psi_m(t)$ for $t \in M$. Fix $t \in (I \backslash M) \cap I^+$. Then by (3.22) there exists an integer n_1 such that

$$y_n(t) \geq \frac{1}{2} A\psi_m(t) \|y_{mn}\|_2, \qquad \text{for } n \geq n_1. \tag{3.30}$$

NOTE. The right-hand side of (3.30) goes to $+\infty$ as $n \to \infty$ in S_2.

Analogously, fixing $t \in (I \backslash M) \cap I^-$, we see from (3.22) that there exists an integer n_2 such that

$$y_n(t) \le \frac{1}{2} A\psi_m(t) \|y_{mn}\|_2, \quad \text{for } n \ge n_2. \tag{3.31}$$

NOTE. The right-hand side of (3.31) goes to $-\infty$ as $n \to \infty$ in S_2.

Now (3.30) and (3.31) in (3.29) imply that

$$\liminf_{n\to\infty} \langle Fy_n, A\psi_m \rangle \ge \int_{I^+} A\psi_m(t) \liminf_{x\to\infty} f(t,x)\,dt + \int_{I^-} A\psi_m(t) \limsup_{x\to -\infty} f(t,x)\,dt,$$

and therefore, (3.24) holds if

$$\begin{gathered}\int_{I^+} A\psi_m(t) \liminf_{x\to\infty} f(t,x)\,dt + \int_{I^-} A\psi_m(t) \limsup_{x\to -\infty} f(t,x)\,dt > 0, \\ \text{where } A = +1 \text{ or } -1, \text{ and } I^+ \text{ and } I^- \text{ are as defined in (3.28).}\end{gathered} \tag{3.32}$$

We have, therefore, just proved the following existence result for (3.1).

THEOREM 3.2. *Suppose $k : [0,T] \times [0,T] \to \mathbf{R}$ satisfies (3.7)–(3.9) and (3.11), and $f : [0,T] \times \mathbf{R} \to \mathbf{R}$ satisfies (3.10), (3.13) with $\alpha = 0$ and (3.32). Then (3.1) has a solution $y \in L^2[0,T]$.*

CASE 2. We now consider (3.13) when $1 > \alpha = \alpha_1/\alpha_2$, with α_1 even and α_2 odd. In fact, the argument in this case will also hold if $\alpha = 0$.

We will obtain a contradiction to (3.23) if we show

$$\liminf_{n\to\infty} \frac{\langle Fy_n, A\psi_m \rangle}{\|y_{mn}\|_2^\alpha} > 0 \tag{3.33}$$

is true. To apply Fatou's Lemma to the above, we must prove that there exists $\phi \in L^1[0,T]$ such that

$$\frac{|f(t, y_n(t))|}{\|y_{mn}\|_2^\alpha} \le \phi(t), \quad \text{a.e. } t \in [0,T] \tag{3.34}$$

holds. To ensure this, we require that there exists a subsequence S_3 of S_1 such that

$$r_n \to A\psi_m \text{ in } C[0,T], \quad \text{as } n \to \infty, \quad n \in S_3. \tag{3.35}$$

Of course for (3.35) to even make sense, we must now strengthen the conditions on $k : [0,T] \times [0,T] \to \mathbf{R}$. In particular, suppose that in addition to (3.9) and (3.11), k also satisfies

$$k_t(s) = k(t,s) \in L^2[0,T], \quad \text{for each } t \in [0,T] \tag{3.36}$$

and

$$\text{the map } t \mapsto k_t \text{ from } [0,T] \text{ to } L^2[0,T] \text{ is continuous,} \tag{3.37}$$

then $K : L^2[0,T] \to C[0,T]$, and for γ not equal to an eigenvalue of K,

$$(I - \gamma K)^{-1} K : L^2[0,T] \to C[0,T]$$

is a bounded, linear, completely continuous operator. (See [4, Remark 2.1.)

Dividing (3.15) by $\|y_{mn}\|_2$, we see that

$$r_n(t) = \delta_n (I - \gamma K)^{-1} K \left[(\lambda_m - \gamma) r_n(t) + \frac{Fy_n(t)}{\|y_{mn}\|_2} \right], \quad t \in [0,T]. \tag{3.38}$$

NOTE. We now have that r_n and ψ_n are in $C[0,T]$, for all n.

Since f satisfies (3.13), there exist constants C_1, $C_2 \geq 0$ such that

$$\|Fy_n\|_2 \leq C_1 + C_2\|y_n\|_2^\alpha.$$

This fact implies that

$$\begin{aligned}\left\|(\lambda_m - \gamma)r_n + \frac{Fy_n}{\|y_{mn}\|_2}\right\|_2 &\leq (\gamma - \lambda_m)\|r_n\|_2 + \frac{C_1 + C_2\|y_n\|_2^\alpha}{\|y_{mn}\|_2} \\ &\leq (\gamma - \lambda_m)\|r_n\|_2 + \frac{C_1}{\|y_{mn}\|_2} + \frac{C_2\|r_n\|_2^\alpha}{\|y_{mn}\|_2^{1-\alpha}},\end{aligned}$$

and thus by (3.21) and (3.18), we have that $(\gamma - \lambda_m)r_n + (Fy_n/\|y_{mn}\|_2)$ is bounded in $L^2[0,T]$. Therefore, since $(I - \gamma K)^{-1}K : L^2[0,T] \to C[0,T]$ is completely continuous, we see from (3.38) that there exists a subsequence S_3 of S_1 such that (3.35) is true. Now from (3.13) we obtain

$$\frac{|f(t, y_n(t))|}{\|y_{mn}\|_2^\alpha} \leq \frac{\phi_1(t)}{\|y_{mn}\|_2^\alpha} + \phi_2(t)|r_n(t)|^\alpha, \qquad \text{a.e. } t \in [0,T],$$

and thus by (3.35), there exists $\phi \in L^1[0,T]$ such that (3.34) is true.

We can apply Fatou's Lemma to $\liminf_{n\to\infty} \langle Fy_n, A\psi_m\rangle / \|y_{mn}\|_2^\alpha$, to obtain for $n \in S_3$,

$$\liminf_{n\to\infty} \frac{\langle Fy_n, A\psi_m\rangle}{\|y_{mn}\|_2^\alpha} \geq \int_{I^+} \liminf_{n\to\infty} \frac{f(t, y_n(t))}{\|y_{mn}\|_2^\alpha} A\psi_m(t)\,dt + \int_{I^-} \liminf_{n\to\infty} \frac{f(t, y_n(t))}{\|y_{mn}\|_2^\alpha} A\psi_m(t)\,dt. \tag{3.39}$$

For any sequences (t_n), recall that if a sequence (s_n) converges to a positive real number s, then $\liminf s_n t_n = s \liminf t_n$. Analogously, if (s_n) converges to a negative real number s, then $\liminf s_n t_n = s \limsup t_n$. This along with (3.35), the fact that $r_n(t) = (y_n(t)/\|y_{mn}\|_2)$ and α is equal to either 0 or α_1/α_2 where α_1 is even and α_2 is odd, implies in (3.39) that for $n \in S_3$,

$$\begin{aligned}&\liminf_{n\to\infty} \frac{\langle Fy_n, A\psi_m\rangle}{\|y_{mn}\|_2^\alpha} \\ &\geq \int_{I^+} \liminf_{n\to\infty} \frac{f(t, y_n(t))}{y_n^\alpha(t)} r_n^\alpha(t) A\psi_m(t)\,dt + \int_{I^-} \liminf_{n\to\infty} \frac{f(t, y_n(t))}{y_n^\alpha(t)} r_n^\alpha(t) A\psi_m(t)\,dt \\ &\geq \int_{I^+} (A\psi_m(t))^{1+\alpha} \liminf_{n\to\infty} \frac{f(t, y_n(t))}{y_n^\alpha(t)}\,dt + \int_{I^-} (A\psi_m(t))^{1+\alpha} \limsup_{n\to\infty} \frac{f(t, y_n(t))}{y_n^\alpha(t)}\,dt.\end{aligned} \tag{3.40}$$

Since $r_n \to A\psi_m$ uniformly in $C[0,T]$ as $n \to \infty$ in S_3, for all $t \in I^+$, there exists n_3 such that for all $n \geq n_3$,

$$y_n(t) \geq \frac{1}{2} A\psi_m(t)\|y_{mn}\|_2, \tag{3.41}$$

and for all $t \in I^-$, there exists n_4 such that for all $n \geq n_4$,

$$y_n(t) \leq \frac{1}{2} A\psi_m(t)\|y_{mn}\|_2. \tag{3.42}$$

NOTE. The right-hand side of (3.41) goes to $+\infty$ as $n \to \infty$, whereas the right-hand side of (3.42) goes to $-\infty$ as $n \to \infty$. Finally, (3.41) and (3.42) imply from (3.40) that

$$\liminf_{n\to\infty} \frac{\langle Fy_n, A\psi_m\rangle}{\|y_{mn}\|_2^\alpha} > 0,$$

if f satisfies

$$\int_{I^+} (A\psi_m(t))^{1+\alpha} \liminf_{x\to\infty} \frac{f(t, x)}{x^\alpha}\,dt + \int_{I^-} (A\psi_m(t))^{1+\alpha} \limsup_{x\to-\infty} \frac{f(t, x)}{x^\alpha}\,dt > 0, \tag{3.43}$$

where $A = +1$ or $A = -1$, and I^+ and I^- are as defined in (3.28).

We have, therefore, proven the following existence result for (3.2).

THEOREM 3.3. *Suppose* $k : [0,T] \times [0,T] \to \mathbf{R}$ *satisfies (3.36), (3.37), (3.9), and (3.11), and* $f : [0,T] \times \mathbf{R} \to \mathbf{R}$ *satisfies (3.10), (3.13) with* $\alpha = 0$ *or* $1 > \alpha = \alpha_1/\alpha_2$ *where* α_1 *is even and* α_2 *is odd, and (3.43). Then (3.2) has at least one solution* $y \in C[0,T]$.

Theorem 3.2 and Theorem 3.3 have the following "dual" versions.

THEOREM 3.4. *Suppose* $k : [0,T] \times [0,T] \to \mathbf{R}$ *satisfies (3.7)–(3.9) and (3.11) with* m *and* $m+1$ *replaced by* $m-1$ *and* m, *respectively, and* $f : [0,T] \times \mathbf{R} \to \mathbf{R}$ *satisfies (3.10), (3.13) with* $\alpha = 0$ *and*

$$\begin{gathered} \int_{I^+} A\psi_m(t) \limsup_{x\to\infty} f(t,x)\,dt + \int_{I^-} A\psi_m(t) \liminf_{x\to-\infty} f(t,x)\,dt < 0, \\ \text{where } A = +1 \text{ or } -1, \text{ and } I^+ \text{ and } I^- \text{ are as defined in (3.28).} \end{gathered} \tag{3.44}$$

Then (3.1) has at least one solution $y \in L^2[0,T]$.

THEOREM 3.5. *Suppose* $k : [0,T] \times [0,T] \to \mathbf{R}$ *satisfies (3.36), (3.37), (3.9), and (3.11) with* m *and* $m+1$ *replaced by* $m-1$ *and* m, *respectively, and* $f : [0,T] \times \mathbf{R} \to \mathbf{R}$ *satisfies (3.10), (3.13) with* $\alpha = 0$ *or* $1 > \alpha = \alpha_1/\alpha_2$ *where* α_1 *is even and* α_2 *is odd, and*

$$\begin{gathered} \int_{I^+} (A\psi_m(t))^{1+\alpha} \limsup_{x\to\infty} \frac{f(t,x)}{x^\alpha}\,dt + \int_{I^-} (A\psi_m(t))^{1+\alpha} \liminf_{x\to-\infty} \frac{f(t,x)}{x^\alpha}\,dt < 0, \\ \text{where } A = +1 \text{ or } A = -1, \text{ and } I^+ \text{ and } I^- \text{ are as defined in (3.28).} \end{gathered} \tag{3.45}$$

Then (3.2) has at least one solution $y \in C[0,T]$.

We omit the details for both theorems as the analysis is similar to that of Theorem 3.2 and Theorem 3.3, respectively.

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